# Rings and ideals

Homework for tonight: Read chapter 1 of A-M and make sure you understand it all.

## Conventions:

- All rings will be commutative with 1.
- Ring homomorphisms send $1 \mapsto 1$.

A big focus in this class will be prime ideals.

**Recall:** An ideal $P \subseteq R$ is prime if whenever $xy \in P$, $x \in P$ or $y \in P$.

Equivalently, $P$ is prime $\Leftrightarrow$ $R/P$ is an integral domain.

$M \subseteq R$ max'l $\Leftrightarrow$ $R/M$ a field.

$u \in R$ is a unit if $uv = 1$, some $v \in R$.

## Nilradical + Jacobson radical

Before moving on, we first introduce some important ideals.

Recall that $x \in R$ is nilpotent if $x^n = 0$ for some $n > 0$.

**Def:** The set $\mathfrak{N}(R)$, or just $\mathfrak{N}$, of all nilpotent elements of $R$ is called the nilradical of $R$.

<u>Check</u>: $\mathfrak{N} \subseteq R$ is an ideal, and $R/\mathfrak{N}$ has no nilpotents.

**Ex**: Consider $k[x,y]/(x^2y)$, $k$ a field. Then $\mathfrak{N} = (xy)$. Note that $x \notin \mathfrak{N}$, $y \notin \mathfrak{N}$, so the nilradical is not necessarily prime.

**Prop**: The nilradical of $R$ is the intersection of all prime ideals in $R$.

<u>**Proof**</u>: Let $\mathfrak{N}' =$ intersection of all prime ideals in $R$.

If $x \in \mathfrak{N}$ and $P \subseteq R$ is a prime ideal, then $x^n = 0 \in P$ for some $n$. So $x \in P$, and thus $\mathfrak{N} \subseteq \mathfrak{N}'$.

If $x \notin \mathfrak{N}$, then $x^n \neq 0$ for any $n$.

Let $\Sigma$ be the set of ideals defined

$$\Sigma = \left\{ I \subseteq R \text{ ideal} \;\middle|\; x^n \notin I \text{ for all } n > 0 \right\}$$

In particular, $0 \in \Sigma$, so $\Sigma$ is nonempty. $\Sigma$ is partially ordered by inclusion.

Let $(a_\alpha)$ be a chain of ideals in $\Sigma$ (i.e. $a_\alpha \subseteq a_\beta$ or $a_\beta \subseteq a_\alpha$ for each $\alpha, \beta$).

Then $\mathfrak{a} = \bigcup \mathfrak{a}_\alpha$ is an ideal (check this) and is thus in $\Sigma$.

So every chain has an upper bound in $\Sigma$, so we can apply Zorn's Lemma* which says $\Sigma$ contains some maximal element $P$.
i.e. $P$ is not properly contained in any element of $\Sigma$.

* Zorn's Lemma says that for any partially ordered sets $S$, if every chain has an upper bound in $S$, then $S$ has a max'l element.

We will show $P$ is prime: Let $y, z \notin P$.
Then $P + (y)$ and $P + (z)$ strictly contain $P$, so are not in $\Sigma$.

Thus, $x^m \in P + (y)$, $x^n \in P + (z)$, some $m, n$.
$\Rightarrow x^{m+n} \in P + (yz)$ (Do you see why?)

Thus, $P + (yz) \not\subset P$, so $yz \notin P$. Thus $P$ is prime.

But $x \notin P$, so $x \notin \mathfrak{N}'$, so $\mathfrak{N}' \subseteq \mathfrak{N}$. $\square$

**Def:** The <u>Jacobson radical</u> of $R$, $J(R)$, is the intersection of all maximal ideals of $R$.

It has the following characterization:

**Prop:** $x \in J(R) \iff 1 - xy$ is a unit for all $y \in R$.

**Pf:** See A–M.

**Def:** The radical of an ideal $I \subseteq R$, denoted $\sqrt{I}$, is defined

$$\sqrt{I} = \{ x \in R \mid x^n \in I, \text{ some } n > 0 \}.$$

Check: $\sqrt{I}$ is an ideal.

Notice that $I \subseteq \sqrt{I}$, and $\sqrt{I}/I = \mathcal{N}(R/I)$. Thus we get:

**Cor:** $\sqrt{I} = \bigcap \{\text{prime ideals containing } I\}$. In particular, $\sqrt{0} = \mathcal{N}(R)$.

(Recall: there is a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{(prime) ideals } J \subseteq R \\ \text{containing } I \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{(prime) ideals} \\ \text{in } R/I \end{array} \right\}$$

$$J \longleftrightarrow J/I \;)$$

## Modules

Homework for tonight: Review sections 1–5 of chapter 2 in A–M (through exact sequences)

Let $M$ be an $R$-module.

**Recall:**

- Ideals are naturally $R$ modules, as are quotients of $R$.
- If $R$ is a field, $M$ is a vector space

The annihilator of $M$ is defined

$$\mathrm{Ann}(M) = \{ r \in R \mid rm = 0 \ \forall \ m \in M \}.$$

Check: $\mathrm{Ann}(M)$ is an ideal of $R$.

$M$ is finitely generated if there is a finite set of generators $x_1, \dots, x_n \in M$ s.t. $M = Rx_1 + \dots + Rx_n$.

$M$ is a free $R$-module if it can be written

$$M = \bigoplus_i M_i$$

where each $M_i \cong R$ (as an $R$-module.)

A f.g. free module is sometimes written $M = R^n$, where $n = \#$ of generators.

**Prop:** $M$ is finitely generated $\Leftrightarrow$ $M$ is isomorphic to a quotient of $R^n$, some $n > 0$.

**Pf:** ($\Rightarrow$) If $M$ is generated by $x_1, \dots, x_n$, then define $\varphi : R^n \to M$ by

$$\varphi(r_1, \ldots, r_n) = r_1 x_1 + \ldots + r_n x_n .$$

Clearly this is a surjective homomorphism of modules, so $M \cong R^n / \ker \varphi$.

($\Leftarrow$) If $M = R^n / N$, then the images of the standard basis elements generate $M$. $\square$

A very important class of R-modules is R-algebras, which are rings that are also R-modules, s.t. the R module preserves the multiplicative structure. That is:

**Def:** Let $R$ and $S$ be rings and $\varphi : R \to S$ a ring homomorphism. Then $S$, equipped w/ the R-module structure induced by $\varphi$, is an R-algebra.

An R-algebra homomorphism is a ring homomorphism that is also an R-module homomorphism.

**Ex:** If $R \subseteq S$ is a subring of $S$, the inclusion gives $S$ the structure of an $R$ algebra.

$S$ is a finitely-generated R-algebra if it can be generated (as a ring) by $\varphi(R)$, $a_1, \ldots, a_n$, where

$a_1, \ldots, a_n \in S$.

In the (common) situation where $R$ is a subring of $S$, we write

$$S = R[a_1, \ldots, a_n].$$

(so in the general setting, $S = \varphi(R)[a_1, \ldots, a_n]$. We often abuse notation and write $R[a_1, \ldots, a_n]$)

Note that we get a natural surjection

$$R[x_1, \ldots, x_n] \twoheadrightarrow R[a_1, \ldots, a_n].$$
$$x_i \longmapsto a_i$$

More generally, $A$ is a f.g. $R$-algebra $\Longleftrightarrow$ there is a surjective $R$-algebra homomorphism

$$R[x_1, \ldots, x_n] \twoheadrightarrow A. \quad \text{(Check this!)}$$

Ex: $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1)$. We write $\mathbb{C} \cong \mathbb{R}[i]$.

Ex: If $k$ is a field and $a_1, \ldots, a_n \in k$, set

$$A = k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)$$

Then $A \cong k[a_1, \ldots, a_n] \cong k$.